

A comparison theorem for backward SPDEs with jumps

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Abstract

In this paper, we obtain a comparison theorem for backward stochastic partial differential equation (SPDEs) with jumps. We apply it to introduce space-dependent convex risk measures as a model for risk in large systems of interacting components.

Key Words: Stochastic partial differential equations (SPDEs), comparison theorem, backward stochastic partial differential equations with jumps, convex risk measures.

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1 Introduction and framework

There are several papers dealing with comparison theorems for backward stochastic partial differential equations (BSPDEs). One of the first seems to be the paper [MY]. The results of that paper were subsequently extended (still for linear BSPDEs only) in the paper [DM]. Other related papers are [DQT] and also our own paper [ØSZ] (for reflected BSPDE).

The paper which seems to be closest to ours is [MYZ]. Here more general non-linear BSPDEs are considered, and a comparison theorem is proved for such equations by exploiting the relation between BSPDEs and coupled systems of forward-backward SDEs (FBSDEs).

Our paper also deals with quite general non-linear BSPDEs, but it differs from [MYZ] in several ways:

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- (i) First, our paper includes jumps.
- (ii) Second, our BSPDEs are slightly different. They have stronger conditions on the second order term, but allow more general drift terms.
- (ii) Third, our method is different, being based on an approximation technique.

Let $B_t = B_t(\omega), t \geq 0$ be a Brownian motion and $\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$ an independent compensated Poisson random measure on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, P)$, where ν is the Lévy measure associated with the Poisson measure $N(\cdot, \cdot)$ on $[0, \infty) \times \mathbb{R}$. Let D be a bounded domain in \mathbb{R}^d . Denote by $A(t)$ the following second order differential operator on D equipped with the Dirichlet boundary condition:

$$A(t) = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} (a_{ij}(t, x) \frac{\partial}{\partial x_j}),$$

where $a = (a_{ij}(t, x)) : [0, T] \times D \rightarrow \mathbb{R}^{d \times d}$ is a measurable, symmetric matrix-valued function. Set $L = L^2(\mathbb{R}, \nu)$. Let $b(t, x, u, v, Z, r(\cdot))$ be a measurable mapping from $[0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times L$ into \mathbb{R} . Let $\beta(t) = (\beta_1(t), \dots, \beta_d(t)), t \geq 0$ be a given progressively measurable \mathbb{R}^d -valued stochastic process. From now on, if $u(t, x)$ is a function of (t, x) , we sometimes write $u(t)$ for the function $u(t, \cdot)$. Consider the solution of the following backward stochastic partial differential equation (BSPDE):

$$\begin{aligned} du(t, x) &= -A(t)u(t)dt - b(t, x, u(t, x), \nabla u(t, x), Z(t, x), r(t, x, \cdot))dt + Z(t, x)dB_t \\ &\quad - \langle \beta(t), \nabla Z(t, x) \rangle dt + \int_{\mathbb{R}} r(t, x, z) \tilde{N}(dt, dz), t \in (0, T) \\ u(T, x) &= \phi(x) \quad a.s. \end{aligned} \tag{1.1}$$

Here ϕ is an \mathcal{F}_T -measurable $H := L^2(D)$ -valued random variable. Let V be the Sobolev space $H_0^{1,2}(D)$ and V^* be its dual.

Definition 1.1 *An adapted random field $u(t, x)$ is said to be the solution of the BSPDE (1.1) if*

- (i) $u \in D([0, T]; H) \cap L^2(\Omega \times [0, T]; V)$,
- (ii) for $t > 0$, u satisfies the following equation almost surely in V^*

$$\begin{aligned} u(t, x) &= \phi(x) + \int_t^T A(s)u(s)ds + \int_t^T b(s, x, u(s, x), \nabla u(s, x), Z(s, x), r(s, x, \cdot))ds \\ &\quad - \int_t^T Z(s, x)dB_s + \int_t^T \langle \beta(s), \nabla Z(s, x) \rangle ds - \int_t^T \int_{\mathbb{R}} r_1(s, x, z) \tilde{N}(ds, dz) \end{aligned} \tag{1.2}$$

Remark. Equation (1.2) is equivalent to that for any $\psi \in V$,

$$\begin{aligned}
\langle u(t, \cdot), \psi \rangle &= \langle \phi(\cdot), \psi \rangle + \int_t^T \langle A(s)u(s), \psi \rangle ds \\
&\quad + \int_t^T \langle b(s, \cdot, u(s, \cdot), \nabla u(s, \cdot), Z(s, \cdot), r(s, \cdot, \cdot)), \psi \rangle ds \\
&\quad - \int_t^T \langle Z_1(s, \cdot), \psi \rangle dB_s - \int_t^T \int_D \langle \beta(s), \nabla \psi(x) \rangle Z(s, x) dx ds \\
&\quad - \int_t^T \int_{\mathbb{R}} \langle r_1(s, \cdot, z), \psi \rangle \tilde{N}(ds, dz)
\end{aligned} \tag{1.3}$$

almost surely.

The aim of this paper is to prove a comparison theorem for the above BSPDEs with jumps.

2 Main result

Introduce the following assumptions:

(A.1). There exists $\delta_1 > 0$ and $0 < a < 1$ such that

$$\sum_{i,j=1}^d a_{ij}(t, x) z_i z_j \geq \left(\frac{1}{2a} |\beta|^2(t) + \delta_1 \right) |z|^2, \quad \forall z \in \mathbb{R}^d \text{ and } x \in D \tag{2.1}$$

(A.2). There exists $C > 0$ such that

$$|b(t, x, u_1, v_1, Z_1, r) - b(t, x, u_2, v_2, Z_2, r)| \leq C(|u_1 - u_2| + |Z_1 - Z_2| + |v_1 - v_2|) \tag{2.2}$$

(A.3).

$$b(t, x, u, v, Z, r_1) - b(t, x, u, v, Z, r_2) \leq \int_{\mathbb{R}} (r_1(z) - r_2(z)) \lambda(t, z) \nu(dz), \tag{2.3}$$

where $0 \leq \lambda(t, z) \leq C(1 \wedge |z|)$.

For $i = 1, 2$, consider BSPDEs:

$$\begin{aligned}
du_i(t, x) &= -A(t)u_i(t)dt - b_i(t, x, u_i(t, x), \nabla u_i(t, x), Z_i(t, x), r_i(t, x, \cdot))dt + Z_i(t, x)dB_t \\
&\quad - \langle \beta(t), \nabla Z_i(t, x) \rangle dt + \int_{\mathbb{R}} r_i(t, x, z) \tilde{N}(dt, dz), \quad t \in (0, T) \\
u_i(T, x) &= \phi_i(x) \quad a.s.,
\end{aligned} \tag{2.4}$$

See e.g. [ØPZ] for information about BSPDEs with jumps.

The following theorem is the main result of this paper.

Theorem 2.1 (*Comparison theorem*) Assume (A.1), (A.2) (A.3) hold for one of the coefficients b_i , say, b_2 . If $\phi_1(x) \leq \phi_2(x)$ and

$$b_1(t, x, u_1(t, x), \nabla u_1(t, x), Z_1(t, x), r_1(t, x, \cdot)) \leq b_2(t, x, u_1(t, x), \nabla u_1(t, x), Z_1(t, x), r_1(t, x, \cdot)),$$

then we have $u_1(t, x) \leq u_2(t, x), x \in D$, a.e. for every $t \in [0, T]$.

Proof. For $n \geq 1$, define functions $\psi_n(z)$, $f_n(x)$ as follows (see [DP1]).

$$\psi_n(z) = \begin{cases} 0 & \text{if } z \leq 0, \\ 2nz & \text{if } 0 \leq z \leq \frac{1}{n}, \\ 2 & \text{if } z > \frac{1}{n}. \end{cases} \quad (2.5)$$

$$f_n(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \int_0^x dy \int_0^y \psi_n(z) dz & \text{if } x > 0. \end{cases} \quad (2.6)$$

We have

$$f'_n(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ nx^2 & \text{if } 0 \leq x \leq \frac{1}{n}, \\ 2x - \frac{1}{n} & \text{if } x > \frac{1}{n}. \end{cases} \quad (2.7)$$

Also $f_n(x) \uparrow (x^+)^2$ as $n \rightarrow \infty$. For $h \in K := L^2(D)$, set

$$F_n(h) = \int_D f_n(h(x)) dx.$$

F_n has the following derivatives for $h_1, h_2 \in K$,

$$F'_n(h)(h_1) = \int_D f'_n(h(x)) h_1(x) dx, \quad (2.8)$$

$$F''_n(h)(h_1, h_2) = \int_D f''_n(h(x)) h_1(x) h_2(x) dx. \quad (2.9)$$

Applying Ito's formula we obtain

$$\begin{aligned}
& F_n(u_1(t) - u_2(t)) \\
&= F_n(\phi_1 - \phi_2) + \int_t^T F'_n(u_1(s) - u_2(s))(A(s)(u_1(s) - u_2(s)))ds \\
&+ \int_t^T F'_n(u_1(s) - u_2(s))(b_1(s, u_1(s), \nabla u_1(s), Z_1(s), r_1(s)) \\
&\quad - b_2(s, u_2(s), \nabla u_2(s), Z_2(s), r_2(s)))ds \\
&+ \int_t^T F'_n(u_1(s) - u_2(s))(\langle \beta(s), \nabla Z_1(s) - \nabla Z_2(s) \rangle)ds \\
&- \int_t^T F'_n(u_1(s) - u_2(s))(Z_1(s) - Z_2(s))dB_s \\
&- \frac{1}{2} \int_t^T F''_n(u_1(s) - u_2(s))(Z_1(s) - Z_2(s), Z_1(s) - Z_2(s))ds \\
&- \int_t^T \int_{\mathbb{R}} \left\{ F_n(u_1(s-) - u_2(s-) + r_1(s, \cdot, z) - r_2(s, \cdot, z)) \right. \\
&\quad \left. - F_n(u_1(s-) - u_2(s-)) \right\} \tilde{N}(ds, dz) \\
&- \int_t^T \int_{\mathbb{R}} \left\{ F_n(u_1(s) - u_2(s) + r_1(s, \cdot, z) - r_2(s, \cdot, z)) - F_n(u_1(s) - u_2(s)) \right. \\
&\quad \left. - F'_n(u_1(s) - u_2(s))(r_1(s, \cdot, z) - r_2(s, \cdot, z)) \right\} ds \nu(dz) \\
&=: I_n^1 + I_n^2 + I_n^3 + I_n^4 + I_n^5 + I_n^6 + I_n^7 + I_n^8. \tag{2.10}
\end{aligned}$$

In view of the assumptions (A.1)–(A.3), we have

$$\begin{aligned}
& I_n^2 = \int_t^T F'_n(u_1(s) - u_2(s))(A(s)(u_1(s) - u_2(s)))ds \\
&= \int_t^T \int_D f'_n(u_1(s, x) - u_2(s, x)) \left(\sum_{i,j=1}^d \frac{\partial}{\partial x_i} (a_{ij}(t, x)) \frac{\partial}{\partial x_j} (u_1(s, x) - u_2(s, x)) \right) dx ds \\
&= - \int_t^T \int_D f''_n(u_1(s, x) - u_2(s, x)) \sum_{i,j=1}^d a_{ij}(t, x) \frac{\partial}{\partial x_i} (u_1(s, x) - u_2(s, x)) \\
&\quad \times \frac{\partial}{\partial x_j} (u_1(s, x) - u_2(s, x)) dx ds, \\
&\leq - \int_t^T \left(\frac{1}{2a} |\beta|^2(s) + \delta_1 \right) ds \int_D f''_n(u_1(s, x) - u_2(s, x)) |\nabla(u_1(s, x) - u_2(s, x))|^2 dx, \tag{2.11}
\end{aligned}$$

$$I_n^6 = -\frac{1}{2} \int_t^T \int_D f_n''(u_1(s, x) - u_2(s, x)) |Z_1(s, x) - Z_2(s, x)|^2 dx ds, \quad (2.12)$$

and

$$\begin{aligned} I_n^8 &= - \int_t^T \int_{\mathbb{R}} \int_D \left\{ f_n(u_1(s, x) - u_2(s, x) + r_1(s, x, z) - r_2(s, x, z)) - f_n(u_1(s, x) - u_2(s, x)) \right. \\ &\quad \left. - f_n'(u_1(s, x) - u_2(s, x))(r_1(s, x, z) - r_2(s, x, z)) \right\} dx ds \nu(dz) \\ &= -\frac{1}{2} \int_t^T \int_{\mathbb{R}} \int_D f_n''(u_1(s, x) - u_2(s, x) + \theta(s, x, z)(r_1(s, x, z) - r_2(s, x, z))) \\ &\quad \times (r_1(s, x, z) - r_2(s, x, z))^2 dx ds \nu(dz), \end{aligned} \quad (2.13)$$

where $0 \leq \theta(s, x, z) \leq 1$.

We further write I_n^3 as

$$\begin{aligned} I_n^3 &= \int_t^T \int_D f_n'(u_1(s, x) - u_2(s, x)) (b_1(s, x, u_1(s, x), \nabla u_1(s, x), Z_1(s, x), r_1(s, x, \cdot)) \\ &\quad - b_2(s, x, u_2(s, x), \nabla u_2(s, x), Z_2(s, x), r_2(s, x, \cdot))) dx ds \\ &= \int_t^T \int_D f_n'(u_1(s, x) - u_2(s, x)) (b_1(s, x, u_1(s, x), \nabla u_1(s, x), Z_1(s, x), r_1(s, x, \cdot)) \\ &\quad - b_2(s, x, u_1(s, x), \nabla u_1(s, x), Z_1(s, x), r_1(s, x, \cdot))) dx ds \\ &\quad + \int_t^T \int_D f_n'(u_1(s, x) - u_2(s, x)) (b_2(s, x, u_1(s, x), \nabla u_1(s, x), Z_1(s, x), r_1(s, x, \cdot)) \\ &\quad - b_2(s, x, u_2(s, x), \nabla u_1(s, x), Z_1(s, x), r_1(s, x, \cdot))) dx ds \\ &\quad + \int_t^T \int_D f_n'(u_1(s, x) - u_2(s, x)) (b_2(s, x, u_2(s, x), \nabla u_1(s, x), Z_1(s, x), r_1(s, x, \cdot)) \\ &\quad - b_2(s, x, u_2(s, x), \nabla u_2(s, x), Z_1(s, x), r_1(s, x, \cdot))) dx ds \\ &\quad + \int_t^T \int_D f_n'(u_1(s, x) - u_2(s, x)) (b_2(s, x, u_2(s, x), \nabla u_2(s, x), Z_1(s, x), r_1(s, x, \cdot)) \\ &\quad - b_2(s, x, u_2(s, x), \nabla u_2(s, x), Z_2(s, x), r_1(s, x, \cdot))) dx ds \\ &\quad + \int_t^T \int_D f_n'(u_1(s, x) - u_2(s, x)) (b_2(s, x, u_2(s, x), \nabla u_2(s, x), Z_2(s, x), r_1(s, x, \cdot)) \\ &\quad - b_2(s, x, u_2(s, x), \nabla u_2(s, x), Z_2(s, x), r_2(s, x, \cdot))) dx ds \\ &:= I_{n,1}^3 + I_{n,2}^3 + I_{n,3}^3 + I_{n,4}^3 + I_{n,5}^3. \end{aligned} \quad (2.14)$$

Now, $I_{n,1}^3 \leq 0$ by the assumption on b_1 and b_2 , and

$$I_{n,2}^3 \leq C \int_t^T \int_D ((u_1(s, x) - u_2(s, x))^+)^2 dx ds, \quad (2.15)$$

by the Lipschitz condition of b_2 . Recalling the constant δ_1 in (A.1), we can find a constant C_{δ_1} such that

$$\begin{aligned} I_{n,3}^3 &\leq C \int_t^T ds \int_D dx f'_n(u_1(s, x) - u_2(s, x)) |\nabla u_1(s, x) - \nabla u_2(s, x)| \\ &\leq \delta_1 \int_t^T ds \int_D f''_n(u_1(s, x) - u_2(s, x)) |\nabla u_1(s, x) - \nabla u_2(s, x)|^2 dx \\ &\quad + C_{\delta_1} \int_t^T ds \int_D \frac{f'_n(u_1(s, x) - u_2(s, x))^2}{f''_n(u_1(s, x) - u_2(s, x))} dx \\ &\leq \delta_1 \int_t^T ds \int_D f''_n(u_1(s, x) - u_2(s, x)) |\nabla u_1(s, x) - \nabla u_2(s, x)|^2 dx \\ &\quad + C \int_t^T \int_D ((u_1(s, x) - u_2(s, x))^+)^2 dx ds, \end{aligned} \quad (2.16)$$

where we have used the fact that there exists a constant C (independent of n) such that for $y \geq 0$,

$$\frac{f'_n(x)^2}{f''_n(x+y)} \leq C(x^+)^2. \quad (2.17)$$

This can be easily checked using the definition of f_n . By a similar trick, for any $\delta_2 > 0$, we have

$$\begin{aligned} I_{n,4}^3 &\leq C \int_t^T \int_D f'_n(u_1(s, x) - u_2(s, x)) |Z_1(s, x) - Z_2(s, x)| dx ds \\ &\leq \delta_2 \int_t^T ds \int_D f''_n(u_1(s, x) - u_2(s, x)) |Z_1(s, x) - Z_2(s, x)|^2 dx \\ &\quad + C_{\delta_2} \int_t^T \int_D ((u_1(s, x) - u_2(s, x))^+)^2 dx ds. \end{aligned} \quad (2.18)$$

In view of the assumption (2.3), we have

$$\begin{aligned}
I_{n,5}^3 &\leq \int_t^T ds \int_D dx f'_n(u_1(s, x) - u_2(s, x)) \int_{\mathbb{R}} (r_1(s, x, z) - r_2(s, x, z)) \lambda(s, z) \nu(dz) \\
&\leq \int_t^T ds \int_D dx f'_n(u_1(s, x) - u_2(s, x)) \\
&\quad \times \int_{\mathbb{R}} (r_1(s, x, z) - r_2(s, x, z)) \chi_{\{r_1(s, x, z) > r_2(s, x, z)\}} \lambda(s, z) \nu(dz) \\
&\leq \frac{1}{2} \int_t^T \int_{\mathbb{R}} \int_D f''_n(u_1(s, x) - u_2(s, x) + \theta(s, x, z)(r_1(s, x, z) - r_2(s, x, z))) \\
&\quad \times (r_1(s, x, z) - r_2(s, x, z))^2 \chi_{\{r_1(s, x, z) > r_2(s, x, z)\}} dx ds \nu(dz) \\
&\quad + C \int_t^T \int_{\mathbb{R}} \lambda(s, z)^2 \int_D \frac{f'_n(u_1(s, x) - u_2(s, x))^2}{f''_n(u_1(s, x) - u_2(s, x) + \theta(s, x, z)(r_1(s, x, z) - r_2(s, x, z)))} \\
&\quad \times \chi_{\{r_1(s, x, z) > r_2(s, x, z)\}} dx ds \nu(dz) \\
&\leq \frac{1}{2} \int_t^T \int_{\mathbb{R}} \int_D f''_n(u_1(s, x) - u_2(s, x) + \theta(s, x, z)(r_1(s, x, z) - r_2(s, x, z))) \\
&\quad \times (r_1(s, x, z) - r_2(s, x, z))^2 dx ds \nu(dz) \\
&\quad + C \int_{\mathbb{R}} (1 \wedge |z|)^2 \nu(dz) \int_t^T \int_D ((u_1(s, x) - u_2(s, x))^+)^2 dx ds, \tag{2.19}
\end{aligned}$$

where we have used (2.17) again.

Now, by integration by parts, for $0 < a < 1$,

$$\begin{aligned}
I_n^4 &= \int_t^T \int_D < f'_n(u_1(s, x) - u_2(s, x)) \beta(s), \nabla(Z_1(s, x) - Z_2(s, x)) > dx ds \\
&= - \int_t^T \int_D < f''_n(u_1(s, x) - u_2(s, x)) \nabla(u_1(s, x) - u_2(s, x)), \\
&\quad \beta(s)(Z_1(s, x) - Z_2(s, x)) > dx ds \\
&\leq \frac{1}{2} a \int_t^T \int_D f''_n(u_1(s, x) - u_2(s, x)) |Z_1(s, x) - Z_2(s, x)|^2 dx ds \\
&\quad + \frac{1}{2a} \int_t^T \beta^2(s) \int_D f''_n(u_1(s, x) - u_2(s, x)) |\nabla(u_1(s, x) - u_2(s, x))|^2 dx ds
\end{aligned}$$

Now, choose $\delta_2 > 0$ sufficiently small to satisfy that

$$\delta_2 + \frac{1}{2}a \leq \frac{1}{2} \tag{2.20}$$

Adding (2.11), (2.13), (2.16), (2.20), (2.14), (2.18) and (2.19) together and taking into account of (2.20) we deduce that

$$I_n^2 + I_n^3 + I_n^4 + I_n^6 + I_n^8 \leq C \int_t^T \int_D ((u_1(s, x) - u_2(s, x))^+)^2 dx ds \tag{2.21}$$

Thus it follows from (2.10), (2.11) and (2.21) that

$$\begin{aligned}
& F_n(u_1(t) - u_2(t)) \\
\leq & F_n(\phi_1 - \phi_2) + C \int_t^T \int_D ((u_1(s, x) - u_2(s, x))^+)^2 dx ds \\
& - \int_t^T F'_n(u_1(s) - u_2(s))(Z_1(s) - Z_2(s)) dB_s \\
& - \int_t^T \int_{\mathbb{R}} \left\{ F_n(u_1(s-) - u_2(s-) + r_1(s, \cdot, z) - r_2(s, \cdot, z)) \right. \\
& \quad \left. - F_n(u_1(s-) - u_2(s-)) \right\} \tilde{N}(ds, dz)
\end{aligned} \tag{2.22}$$

Take expectation and let $n \rightarrow \infty$ to get

$$E\left[\int_D ((u_1(t, x) - u_2(t, x))^+)^2 dx\right] \leq \int_t^T ds E\left[\int_D ((u_1(s, x) - u_2(s, x))^+)^2 dx\right] \tag{2.23}$$

Gronwall's inequality yields that

$$E\left[\int_D ((u_1(t, x) - u_2(t, x))^+)^2 dx\right] = 0, \tag{2.24}$$

which completes the proof of the theorem. \blacksquare

3 Application

Let $u(t, x)$ be the solution of a BSDE of the form (1.1) satisfying the conditions (A.1)-(A.3). Assume that b does not depend on u , i.e.

$$b(t, x, u, \nabla u, Z, r) = b(t, x, Z, r) \quad \text{for all } t, x, u, Z, r. \tag{3.1}$$

Moreover, assume that $b(t, x, Z, r)$ is concave with respect to Z, r for all t, x . If we, for example, regard $\phi(x)$ as a financial standing at time $t = T$ and at the point x , we may as in [ØSZ] define the risk $\rho(\phi)(x)$ of ϕ at time $t = 0$ and at the point x by

$$\rho(\phi)(x) = -u(0, x); \quad x \in \mathbb{R}^d. \tag{3.2}$$

Using the comparison theorem (Theorem 2.1) we can now verify that $\phi \rightarrow \rho(\phi)$ is a convex risk measure, in the sense that it satisfies the following conditions:

(3.3) (Convexity) $\rho(\lambda\phi_1 + (1 - \lambda)\phi_2) \leq \lambda\rho(\phi_1) + (1 - \lambda)\rho(\phi_2)$ for all $\lambda \in [0, 1]$ and all ϕ_1, ϕ_2 .

(3.4) (Monotonicity) $\phi_1 \leq \phi_2 \Rightarrow \rho(\phi_1) \geq \rho(\phi_2)$.

(3.5) (Translation invariance) $\rho(\phi + a) = \rho(\phi) - a$ for all ϕ and all constants a .

Thus we have an extension of the convex risk measure concept (see e.g. [FS]) to a space-dependent situation. This might be of relevance in large systems of interacting components.

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